The Mathematics Of Stock Option Valuation - Part Four
Deriving The Black-Scholes Model Via Partial Differential Equations

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October 2010

In Part One we explained why valuing a call option as a stand-alone asset using risk-adjusted discount rates will almost always lead to an incorrect value because the value determined in this manner will most likely be subject to arbitrage. In Part Two we calculated the no-arbitrage price of a call option in the one-period economy via partial differential equations. In Part III we calculated the no-arbitrage price of a call option in the one-period economy via risk-neutral probabilities. In this section we will switch from the one-period economy to the multi-period economy in continuous time where we will derive the Black-Scholes option pricing model via partial differential equations.

The One Period Economy (From Part One)

The continous time equivalent to our one-period economy in Parts II and III is...

Table 1: Muti-Period Continuous Time Economy

Stock price at time zero	S_0	\$100.00
Call option exercise price	K	\$120.00
Annual discount rate	μ	0.30
Annual return volatility	σ	0.50
Annual risk-free rate	r	0.05
Time to option expiration in years	T	1.00

In Part One we estimated the stock price at time $t = 0$ to be \$100.00 using a discount rate of 30% and an approximate volatility of 50%. We currently sit at time $t = 0$ where the state-of-the-world at time $t = 1$ is unknown. In this section we will derive the Black-Scholes equation and use that equation to calculate the no-arbitrage value of the call option at time $t = 0$.

Legend of Symbols

C_t = Call option price at the end of time t
P_t = Hedge portfolio value at the end of time t
S_t = Stock price at the end of time t
Δ_t = Number of shares of stock in the hedge portfolio at time t
K = Call option exercise price
T = Time to option expiration in years
r = Annual risk-free rate of interest
t = Current time period in years
μ = Annual expected return on the stock
σ = Annual standard deviation of returns (Volatility)
W_t = Brownian motion with mean zero and variance t

A Continuous Time Model For Stock Price

We will model stock price as a continuous time stochastic process. The equation for stock price at time t as a function of a deterministic return and an innovation is...

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{1}$$

Per this equation stock price is a function of drift, which is represented by $(\mu - \frac{1}{2}\sigma^2)t$, and a Brownian motion, which is represented by σW_t. The discount rate used to value our stock at time $t = 0$ was 30% so therefore the variable μ in the equation above is 0.30. Drift is fully predictable in that we expect the stock to earn at an annual rate of 30%. The annual return volatility for our stock is 50% and therefore the variable σ in the equation above is 0.50. Sitting at time $t = 0$ we don't know the value of the Brownian motion W_t but we do know that it is normally-distributed with mean zero and variance t. The continuous time equation for our stock price at any time t is...

$$S_t = 100.00\, e^{(0.30 - \frac{1}{2}0.50^2)t + 0.50 W_t} \tag{2}$$

Stock price equation (1) is once differentiable with respect to time and twice differentiable with respect to the Brownian motion. The equation for the total change in stock price (referred to as a stochastic differential equation (SDE)) is...

$$\begin{aligned}\delta S_t &= \frac{\delta S_t}{\delta t}\delta t + \frac{\delta S_t}{\delta W_t}\delta W_t + \frac{1}{2}\frac{\delta^2 S_t}{\delta W_t^2}\delta W_t^2 \\ &= S_t \mu \delta t + S_t \sigma \delta W_t \end{aligned} \tag{3}$$

We can view δt^2 as the variance of δt and δW_t^2 as the variance of δW_t. Time is a deterministic variable and therefore the variance of the change in time is zero, which means that the second derivative of equation (1) with respect to time is zero. The Brownian motion is a random variable and therefore the variance of the change in the Brownian motion is non-zero, which means that the second derivative of equation (1) with respect to the Brownian motion is non-zero. By definition...

$$\mathbb{E}\Bigg[\delta t^2\Bigg] = 0 \;\; ; \;\; \mathbb{E}\Bigg[\delta t\, \delta W_t\Bigg] = 0 \;\; ; \;\; \mathbb{E}\Bigg[\delta W_t^2\Bigg] = \delta t \tag{4}$$

For the partial differential equation developed below we will need an equation for the square of the change in stock price which is...

$$\begin{aligned}\delta S_t^2 &= (S_t \mu \delta t + S_t \sigma \delta W_t)^2 \\ &= S_t^2 \mu^2 \delta t^2 + 2 S_t^2 \mu \sigma \delta t \delta W_t + S_t^2 \sigma^2 \delta W_t^2 \end{aligned} \tag{5}$$

After noting the definitions in equation (4) above we can rewrite equation (5) as...

$$\begin{aligned}\delta S_t^2 &= S_t^2 \mu^2 (0) + 2 S_t^2 \mu \sigma (0) + S_t^2 \sigma^2 (\delta t) \\ &= S_t^2 \sigma^2 \delta t \end{aligned} \tag{6}$$

A Continuous Time Model For Call Price

Our task is to derive an equation for call price at $t = 0$ and although we do not know the exact equation we do know its general form. We will model call price as a function of time (t) and stock price (S_t). The general form of the equation for call price is...

$$C_t = C(S_t, t) \tag{7}$$

Call price is once differentiable with respect to time and twice differentiable with respect to stock price. The equation for the total change in call price at time t is...

$$\delta C_t = \frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\delta S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\delta S_t^2 \tag{8}$$

We can substitute equation (3) for δS_t and equation (6) for δS_t^2 in the equation above. After making these substitutions equation (8) becomes...

$$\delta C_t = \frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\Bigg\{\mu S_t \delta t + \sigma S_t \delta W_t\Bigg\} + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\Bigg\{\sigma^2 S_t^2 \delta t\Bigg\} \tag{9}$$

The equation for the total change in the discounted call price at time t is...

$$\begin{aligned}\delta(e^{-rt} C_t) &= (\delta e^{-rt}) C_t + (\delta C_t) e^{-rt} \\ &= -r e^{-rt} C_t \delta t + e^{-rt}\Bigg[\frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\Bigg\{\mu S_t \delta t + \sigma S_t \delta W_t\Bigg\} + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\Bigg\{\sigma^2 S_t^2 \delta t\Bigg\}\Bigg] \\ &= e^{-rt}\Bigg[- r C_t + \frac{\delta C_t}{\delta t} + \frac{\delta C_t}{\delta S_t}\mu S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2\Bigg]\delta t + e^{-rt}\frac{\delta C_t}{\delta S_t}\sigma S_t \delta W_t \end{aligned} \tag{10}$$

A Continuous Time Model For The Hedge Portfolio

The hedge portfolio will consist of a long position in shares of the underlying stock and a position in a money market account. The value of the hedge portfolio at any time t is...

$$\begin{aligned} X_t &= Stock + Money\,Market \\ &= \Delta_t S_t + (X_t - \Delta_t S_t) \end{aligned} \tag{11}$$

The equation for the total change in value of the hedge portfolio at time t is...

$$\begin{aligned} \delta X_t &= \Delta_t \delta S_t + r(X_t - \Delta_t S_t)\delta t \\ &= \Delta_t(\mu S_t \delta t + \sigma S_t \delta W_t) + r(X_t - \Delta_t S_t)\delta t \\ &= rX_t\delta t + \Delta_t(\mu - r)S_t\delta t + \Delta_t \sigma S_t \delta W_t \end{aligned} \tag{12}$$

The equation for the total change in discounted value of the hedge portfolio at time t is...

$$\begin{aligned} \delta(e^{-rt}X_t) &= (\delta e^{-rt})X_t + (\delta X_t)e^{-rt} \\ &= (-re^{-rt}\delta t)X_t + (rX_t\delta t + \Delta_t(\mu - r)S_t\delta t + \Delta_t\sigma S_t\delta W_t)e^{-rt} \\ &= -re^{-rt}X_t\delta t + re^{-rt}X_t\delta t + \Delta_t(\mu - r)e^{-rt}S_t\delta t + \Delta_t\sigma e^{-rt}S_t\delta W_t \\ &= \Delta_t(\mu - r)e^{-rt}S_t\delta t + \Delta_t\sigma e^{-rt}S_t\delta W_t \end{aligned} \tag{13}$$

Pricing Derivatives Via PDEs

Since both the stock and the call option on that stock are driven by the same random process (i.e. the Brownian motion W_t) these two assets can be combined in one portfolio such that the randomness of one asset offsets the randomness of the other resulting a portfolio that is risk-free. The price of the call option will be determined via the following steps...

1 Create the hedge portfolio and derive the PDE
2 Find the equation that solves the PDE derived in step 1
3 Use the equation in step 2 to determine call price at $t = 0$

We will follow these steps to price the call option in our one period economy.

Step One - Create The Hedge Portfolio And Derive The PDE

We can hedge a short position in a call option via a hedging portfolio that starts with some initial capital X_0 and invests in the underlying stock and a money market account. The goal of the hedging strategy is to have the hedge portfolio value X_t equal to the call option value C_t at every time t. The goal of the hedging strategy in equation form is...

$$X_t = C_t \;\; for \;\; all \;\; t \in [0, T] \tag{14}$$

The combination of the hedge portfolio and the short position in the call is risk-free and therefore the present value at time $t = 0$ of the hedge portfolio and the call option is the value of the hedge portfolio and the call option at time $t > 0$ discounted at the risk-free rate. The equation for the present value of the hedge portfolio and the call option at time $t = 0$ is...

$$e^{-rt}X_t = e^{-rt}C_t \;\; for \;\; all \;\; t \in [0, T] \tag{15}$$

The equivalent of equation (15) above is...

$$X_0 + \int_0^t \delta(e^{-ru}X_u) = C_0 + \int_0^t \delta(e^{-ru}C_u) \tag{16}$$

The amount of capital that we will deposit into the hedge at time $t = 0$ is X_0, which we will define as being equal to C_0. Because $X_0 = C_0$ we can subtract X_0 from the left side and C_0 from the right side of equation (16) above. The revised equation is...

$$\int_0^t \delta(e^{-ru}X_u) = \int_0^t \delta(e^{-ru}C_u) \tag{17}$$

Per equation (13) the hedge is self-financing after the initial capital contribution (i.e. once the hedge is set up no cash is deposited to the hedge or withdrawn from the hedge prior to call expiration). What we need is an equation for call value such that $\delta(e^{-rt}X_t)$ in equation (16) is equal to $\delta(e^{-rt}C_t)$ in equation (16) at every time t. This relationship in mathematical form is...

$$\delta(e^{-rt}X_t) = \delta(e^{-rt}C_t) \;\; for \;\; all \;\; t \in [0,T] \tag{18}$$

We start by substituting equation (13) for $\delta(e^{-rt}X_t)$ and equation (9) for $\delta(e^{-rt}C_t)$ in equation (18) above such that the equation becomes...

$$\Delta_t(\mu - r)e^{-rt}S_t\delta t + \Delta_t \sigma e^{-rt}S_t\delta W_t = e^{-rt}\Bigg[-rC_t + \frac{\delta C_t}{\delta t} + \frac{\delta C_t}{\delta S_t}\mu S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2\Bigg]\delta t + e^{-rt}\frac{\delta C_t}{\delta S_t}\sigma S_t \delta W_t \tag{19}$$

We then multiply both sides of equation (19) by e^{rt} such that the equation becomes...

$$\Delta_t(\mu - r)S_t\delta t + \Delta_t \sigma S_t\delta W_t = \Bigg[-rC_t + \frac{\delta C_t}{\delta t} + \frac{\delta C_t}{\delta S_t}\mu S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2\Bigg]\delta t + \frac{\delta C_t}{\delta S_t}\sigma S_t \delta W_t \tag{20}$$

We want to remove all randomness from equation (20) by eliminating all terms that involve δW_t. We do this by setting the number of shares of the underlying stock held by the hedge portfolio equal to the first derivative of call price with respect to stock price. We will make the following definition...

$$\Delta_t = \frac{\delta C_t}{\delta S_t} \tag{21}$$

After making this substitution a portfolio that is long the hedge portfolio and short the call option is risk-free. Equation (20) becomes...

$$\begin{aligned}
\frac{\delta C_t}{\delta S_t}(\mu - r)S_t\delta t + \frac{\delta C_t}{\delta S_t}\sigma S_t\delta W_t &= \Bigg[-rC_t + \frac{\delta C_t}{\delta t} + \frac{\delta C_t}{\delta S_t}\mu S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2\Bigg]\delta t + \frac{\delta C_t}{\delta S_t}\sigma S_t \delta W_t \\
-r\frac{\delta C_t}{\delta S_t}S_t\delta t &= \Bigg[-rC_t + \frac{\delta C_t}{\delta t} + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2\Bigg]\delta t \\
0 &= \Bigg[-rC_t + \frac{\delta C_t}{\delta t} + r\frac{\delta C_t}{\delta S_t}S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2\Bigg]\delta t
\end{aligned} \tag{22}$$

Since δt is common to all terms in equation (22) we can remove it. Equation (22) above becomes the Black-Scholes partial differential equation...

$$-rC_t + \frac{\delta C_t}{\delta t} + r\frac{\delta C_t}{\delta S_t}S_t + \frac{1}{2}\frac{\delta^2 C_t}{\delta S_t^2}\sigma^2 S_t^2 = 0 \tag{23}$$

Step Two - Find A Solution To The PDE

The solution to a partial differential equation is another equation such that when you take the solution equation derivatives and drop them into equation (23) above you get zero. Our PDE has an infinite number of solutions. To get one unique solution we must specify boundary conditions. We will add the condition that the value of the call at expiration (time T) must be equal to...

$$C_{T,S_T} = Max(S_T - K, 0) \tag{24}$$

As it turns out the Black-Scholes PDE is the one-dimensional heat equation in disguise. Rather than solving the Black-Scholes PDE via the heat equation we will prove that the solution to the PDE is indeed valid. The solution to the PDE via the one-dimensional heat equation and subject to the boundary conditions in equation (24) above is...

$$C_0 = S_0 N(d1) - Ke^{-r(T-t)}N(d2) \tag{25}$$

where the equation for d1 is...

$$d1 = \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{26}$$

and the equation for d2 is...

$$d2 = d1 - \sigma\sqrt{T-t} \tag{27}$$

In order to prove that equation (25) is a valid solution to the PDE in equation (23) we need the solution equation's derivatives. We can obtain "The Greeks" from any decent textbook on the Black-Scholes model and indeed that is what we have done. The derivatives of equation (25) with respect to time (Theta), stock price (Delta) and the square of stock price (Gamma) are...

Delta...

$$\frac{\delta C}{\delta S} = N(d1) \tag{28}$$

Gamma...

$$\frac{\delta^2 C}{\delta S^2} = \frac{N'(d1)}{S\sigma\sqrt{t}} \tag{29}$$

Theta...

$$\frac{\delta C}{\delta t} = -\frac{SN'(d1)\sigma}{2\sqrt{t}} - rKe^{-rt}N(d2) \tag{30}$$

We will now drop equations (28), (29) and (30) into equation (23)...

$$\begin{aligned} -rC_t + \frac{\delta C_t}{\delta t} + rS_t\frac{\delta C_t}{\delta S_t} + \frac{1}{2}S_t^2\sigma^2\frac{\delta^2 C_t}{\delta S_t^2} &= 0 \\ -r\Big\{S_tN(d1) - Ke^{-rt}N(d2)\Big\} + \Big\{-\frac{SN'(d1)\sigma}{2\sqrt{t}} - rKe^{-rt}N(d2)\Big\} + rS_t\Big\{N(d1)\Big\} + \frac{1}{2}S_t^2\sigma^2\Big\{\frac{N'(d1)}{S\sigma\sqrt{t}}\Big\} &= 0 \\ -rS_tN(d1) + rKe^{-rt}N(d2) - \frac{SN'(d1)\sigma}{2\sqrt{t}} - rKe^{-rt}N(d2) + rS_tN(d1) + \frac{S_tN'(d1)\sigma}{2\sqrt{t}} &= 0 \end{aligned} \tag{31}$$

Conclusion: Equation (25) is a solution to PDE equation (23).

Step Three - Determine Call Price At Time Zero

We will now solve for call price at time $t = 0$. The value of d1 is...

$$d1 = \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\frac{100.00}{120.00} + (0.05 + \frac{1}{2}0.50^2)(1-0)}{0.50\sqrt{1-0}} = -0.01464 \tag{32}$$

The value of d2 is...

$$d2 = d1 - \sigma\sqrt{T-t} = -0.01464 - (0.50)(1) = -0.51464 \tag{33}$$

The cumulative normal distribution value of d1 is...

$$N(d1) = N(-0.01464) = 0.49416 \tag{34}$$

The cumulative normal distribution value of d2 is...

$$N(d2) = N(-0.51464) = 0.30340 \tag{35}$$

The value of the call option at time $t = 0$ is...

$$\begin{aligned} C_0 &= S_0N(d1) - Ke^{-r(T-t)}N(d2) \\ &= (100.00)(0.49416) - (120.00)(0.95122)(0.30340) \\ &= 14.78 \end{aligned} \tag{36}$$

Conclusion

The value of the call option at time $t = 0$ via the continuous time Black-Scholes equation is \$14.78. The value of the call option at time $t = 0$ using the discrete time single time step equations in Part II and Part III was \$14.34 and \$14.29, respectively.